ON RING EXTENSIONS FOR COMPLETELY PRIMARY NONCOMMUTATIVE RINGS

BY

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0. **Introduction.** It is the authors' purpose in this paper to initiate the study of ring extensions for completely N primary noncommutative rings which satisfy the ascending chain condition for right ideals (A.C.C.). We begin here by showing that every completely N primary ring R with A.C.C. is properly contained in just such a ring. This is accomplished by first showing that $R[x]$, x an indeterminate where $ax = xa$ for all $a \in R$, is N primary and then constructing the right quotient ring $Q(R[x])$. The details of these results appear in §§ 1, 7 and 8. The corresponding results for the commutative case are given by E. Snapper in [7] and [8].

If $R \subset A$, where A is completely N primary with A.C.C. then,from the discussion in the preceding paragraph, it would seem natural to examine the structure of $R(\sigma)$ when $\sigma \in A$ and $a\sigma = \sigma a$ for all $a \in R$ in the cases where σ is algebraic or transcendental over R. These structures are determined in §§ 6 and 8 of the present paper.

The definitions and notations given in [2] will be used throughout this paper. As in [2], for a ring R, N or $N(R)$ denotes the union of nilpotent ideals([1]) of R, P or $P(R)$ denotes the set of nilpotent elements of R and J or $J(R)$ the Jacobson radical of R. The letter H is used for the natural homomorphism from R to $R/N = \bar{R}$. If B is a subset of R then $\bar{B}$ denotes the image of B under H. If $N = P$ in R and if R' is a ring contained in R then $N(R') = N \cap R'$ and $\bar{R}' = R'/N(R')$. Thus we consider the contraction of H on R' as the natural homomorphism from R' onto $R'/N(R')$.

Unlike the commutative case, the results of this paper will at times depend on the three conditions (i), (ii) and (iii) of [2, § 3]. Therefore, we make the following definition.

DEFINITION 0.1. A ring R with identity is called an *extendable ring* if it satisfies the three conditions:

(i) $P(q)$ is an ideal when q is a right P primary ideal([2]).

(ii) $P(R) = N(R)$.

(iii) The nontrivial completely prime ideals of R/N are maximal right ideals.

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(1) Ideal shall always mean two-sided ideal.

(2) See [2, §1] for the meaning of $P(q)$.

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1. **Properties of $R[x]$.** In [2] we defined a ring R to be N primary if $ab=0$, $a\neq 0$ implies $b\in N$ and $b\neq 0$ implies $a\in N$. A ring R with identity is completely N primary if R/N is a division ring. Note that if R is N primary or completely N primary then N is a completely prime ideal and $N=P$. For the rings considered in this paper N and P shall always be equal.

THEOREM 1.1. *If R is N primary then, in the polynomial ring $R[x]$, $N[x]=P[x]=N(R[x])=P(R[x])=J(R[x])$.*

Proof. We may write $R[x]/N[x]\cong(R/N)[x]$ where $N[x]\subseteq J(R[x])$. Since R/N is an integral domain it contains no nonzero nil ideals and hence, from Theorem 4 of [6, p. 12], $R[x]/N[x]$ is semisimple. Thus $J(R[x])=N[x]$.

Certainly $N[x]=P[x]\subseteq P(R[x])$. We now show that $P(R[x])\subseteq P[x]$. If $f\in P(R[x])$ then $f^n=0$ for some positive integer n. Hence in $R[x]/P[x]\cong(R/P)[x]$ we have $(\tilde{f})^n=\tilde{0}$(3). Since $(R/P)[x]$ has no divisors of zero it follows that $f\in P[x]$.

Next we show that $N(R[x])=N[x]$. Clearly $N(R[x])\subseteq P(R[x])=N[x]$. If $f\in N[x]$ then $f=a_nx^n+\ldots+a_1x+a_0$ for $a_i\in q_i$ where q_i is an ideal of R and $q_i^{t_i}=0$ for positive integers t_i, $i=0, 1, \ldots, n$. Thus f is contained in the ideal $k=q_0[x]+q_1[x]+\ldots+q_n[x]$ and $k^{t_0+t_1+\cdots+t_n+1}=0$. Hence $f\in N(R[x])$. This completes the proof.

As a consequence of this theorem we have, for an N primary ring R, that $R[x]/N(R[x])=R[x]/N[x]\cong(R/N)[x]$. Thus we can consider $(R[x])^-$ as $\bar{R}[x]$. In this case, the natural homomorphism H from $R[x]$ onto $\bar{R}[x]$ maps the polynomial $\sum a_ix^i$ on $\sum\bar{a}_ix^i$.

THEOREM 1.2. *Let R be an N primary ring. Then $f=a_nx^n+\ldots+a_0$ is a unit of $R[x]$ if and only if a_0 is a unit of R and $a_1,\ldots,a_n\in N$.*

Proof. If f is of this form then $\bar{f}$ is a unit of $\bar{R}[x]$. Hence by 2a of [2] f is a unit of $R[x]$.

Conversely, if $f=a_nx^n+\ldots+a_0$ is a unit of $R[x]$ then $\bar{f}=\bar{a}_nx^n+\ldots+\bar{a}_0$ is a unit of $\bar{R}[x]$. But since $\bar{R}$ is an integral domain it follows that $\bar{f}=\bar{a}_0$ is a unit of $\bar{R}$ and $\bar{a}_1=\ldots=\bar{a}_n=\bar{0}$. Thus a_0 is a unit of R and $a_1,\ldots,a_n\in N(R)$.

THEOREM 1.3. *If R is completely N primary and $N^n=0$ for some positive integer n, then $R[x]$ is $N[x]$ primary.*

Proof. This proof will be by induction on the smallest integer n such that $N^n=0$. If $n=1$ obviously $R[x]$ is $N(R[x])$ primary. By Theorem 1.1 $N(R[x])=N[x]$ and thus $R[x]$ is $N[x]$ primary.

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(3) Here the symbol $\sim$ denotes the coset modulo $P[x]$.

Suppose the theorem is true for all rings R where $N^n = 0$, $N^{n-1} \neq 0$. Let R satisfy the conditions of the theorem and $N^{n+1} = 0$, $N^n \neq 0$. Consider the ring R/N^n. The radical of this ring is N/N^n and $(N/N^n)^n$ is the zero coset. In addition since $(R/N^n)/(N/N^n) \cong R/N$ we have that R/N^n is completely N/N^n primary. Hence, by the induction hypothesis $(R/N^n)[x]$ is $(N/N^n)[x]$ primary.

Suppose in $R[x]$ we have $fg = 0$, $g \neq 0$ and $f \notin N[x]$. Since $(R/N)[x]$ is an integral domain $g \in N[x]$. If $g \notin N^n[x]$, then $\tilde{f}\tilde{g} = \tilde{0}$ in $(R/N^n)[x]$ and from above $\tilde{f} \in (N/N^n)[x]$. Thus $f \in N[x]$, a contradiction. Suppose $g \in N^n[x]$. Let a_i be the coefficient of the highest power of x in f which is not in N. Since $N^{n+1} = 0$ and $fg = 0$, we have $a_i b_s = 0$ where $g = b_s x^s + \ldots + b_0, b_s \neq 0$. Since R is completely N primary $a_i \in N$. Thus in any case the assumption that $fg = 0, g \neq 0, f \notin N[x]$ leads to a contradiction. Hence if $g \neq 0$ we can only conclude that $f \in N[x]$. Similarly if $gf = 0$ and $f \neq 0$ then $g \in N[x]$. Therefore, $R[x]$ is $N[x]$ primary and the proof by induction is complete.

2. **Extensions of rings.** A ring A is an *extension* of a ring R if $R \subseteq A$. In the remainder of this paper, unless otherwise stated, we assume that if A is an extension of R then R and A have the same identity element. If $R \subseteq A$ then A can be considered as an R module([4]) with submodule R. Thus, in order to develop a theory for extensions of a ring, it is convenient to discuss some notions concerning modules.

Let R be a ring with identity and M a unital R module. If R_1 and M_1 are nonempty subsets of R and M respectively, then R_1M_1 denotes the set of all finite sums $\sum r_i m_i$ where $r_i \in R_i$ and $m_i \in M_1$. A subset S of M is called a *generating* system of a submodule Q of M if $Q = RS$. If S contains a finite number of elements then Q is said to be *finitely generated.* A finite generating system S of Q is called a *basis* of Q if Q does not have a generating system containing fewer elements than S. If Q is finitely generated then the *rank* $\rho(Q)$ of Q is the number of elements in a basis of Q.

The following theorem was proved in [7, p. 685] for commutative rings. The proof carries over immediately to the noncommutative case.

THEOREM 2.1. *Let M be an R module and Q a submodule of M such that $M - Q$ is finitely generated. If q is any J ideal of R and Q' any submodule of M let $\tilde{S}$ denote the image of the subset S of M under the natural homomorphism from M onto $M - qQ'$. Then $M = Q + RS$ if and only if $\tilde{M} = \tilde{Q} + R\tilde{S}$. Consequently $\rho(M - Q) = \rho(\tilde{M} - \tilde{Q})$ and if $\tilde{M} = \tilde{Q}$ then $M = Q$.*

If a ring A has finite rank as a module over a subring R we call the rank the *degree* $[A : R]$ of the ring extension and say that A is a finite extension of R.

([4]) R module shall always mean left R module. For a discussion of left R modules read Chapter I of [6].

THEOREM 2.2. *Let R and A be rings where $R \subseteq A$ and suppose every right unit of A is a left unit. Then $R = A$ if and only if $[A:R] = 1$.*

Proof. If $R = A$ then the identity element of A is a basis of A and hence $[A:R] = 1$. Conversely if $[A:R] = 1$, let z be a basis of A. Then since $1 \in A$ we have $az = 1$ for some $a \in R$. Thus z is a right and left unit of A and certainly regular. Then for $z^2 \in A$ we have $bz = z^2$ for some $b \in R$ which implies that $b = z \in R$. Thus $R = A$.

By statements 1.1 and 1.2 of [2] it follows that the conditions of Theorem 2.2 are satisfied when A is a J primary ring. In addition, if R and A have the same identity element and A/N is a principal ideal domain, then by the discussion in §5 of [2] we know that Theorem 2.2 is valid.

If R and A are rings where $R \subseteq A$, the *contraction* q_* of an ideal q of A is defined as the largest ideal of R which is contained in q, i.e., $q_* = q \cap R$. The *extension* q^* of an ideal q of R in A is defined as the smallest ideal of A which contains q, i.e., $q = AqA$, the set of all sums $\sum a_i q_i b_i$ where $q_i \in q$ and $a_i, b_i \in A$. Thus, by definition, $N(R)^* = AN(R)A$. Of particular importance to us is the case when $N(R)^* = N(R)A$ and in addition $N(R)^* = N(A)$. We make the following

DEFINITION 2.1. A ring A is called a *principal extension* of a subring R if $N(A) = N(R)^* = N(R)A$.

If R is N primary then, by Theorem 1.1, the ring $R[x]$ is a principal extension of R.

THEOREM 2.3. *Let A be a finite principal extension of R. If S is a subset of A then $A = RS$ if and only if $\bar{A} = \bar{R}\bar{S}$. Hence $[A:R] = [\bar{A}:\bar{R}]$ and if $\bar{R} = \bar{A}$ then $R = A$.*

Proof. If A is a principal extension of R then $N(A) = N(R)A$. Consider A as an R module with $N(R) \subseteq J(R)$. The first part of the theorem then follows from Theorem 2.1 by setting $A = Q' = M$, $Q = 0$ and $q = N(R)$. If in Theorem 2.1 we let $A = Q' = M, Q = R$ and $q = N(R)$ it follows that $\bar{R} = \bar{A}$ implies $R = A$.

THEOREM 2.4. *Let $R = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n$, where A_i is a finite principal extension of A_{i-1} of degree $[A_i:A_{i-1}] = r_i$ for $i = 1, 2, \ldots, n$. Then, if all the rings $R, A_1, \ldots, A_{n-1}$ are completely N primary, $[A_n:R] = r_1 r_2 \ldots r_n$.*

Proof. Since $N(A_n) = N(R)A_1A_2 \ldots A_n = N(R)A_n$ it follows that A_n is a finite principal extension of R. Hence $[A_n:R] = [\bar{A}_n:\bar{R}]$. Since A_i is a finite principal extension of A_{i-1}, $r_i = [\bar{A}_i:\bar{A}_{i-1}]$. Consequently, since $\bar{R}, \bar{A}_1, \ldots, \bar{A}_{n-1}$ are division rings we have $[A_n:R] = r_1 r_2 \ldots r_n$.

3. Degrees of ideals.

DEFINITION 3.1. Let R and A be rings where $R \subseteq A$. An ideal p of A has *finite degree* $\deg(p)$ if the ring A/p is a finite extension of R/p_*. If p has finite degree then $\deg(p) = [A/p:R/p_*]$.

Definition 3.1 is equivalent to saying that if the rank of the R module $A - p$ is finite then $\deg(p) = \rho(A - p)$.

THEOREM 3.1. *Let A be a principal extension of R and p an ideal of A of finite degree. If S is a subset of A then $A = p + RS$ if and only if $\bar{A} = \bar{p} + \bar{R}\bar{S}$. Hence* $\deg(p) = \deg(\bar{p})$.

Proof. Apply Theorem 2.1 with $M = Q' = A$, $Q = p$ and $q = N(R)$.

The following two results are proved in [7] for commutative rings. Using the definitions and results listed above the proofs now carry over, without essential modification, to the noncommutative case.

STATEMENT 3.1. An ideal p of $R[x]$ has finite degree if and only if p contains a monic polynomial([5]).

THEOREM 3.2. *If R is N primary, then an ideal p of $R[x]$ has finite degree if and only if $\bar{p}$ has finite degree in $\bar{R}[x]$. In this case if B is a subset of $R[x]$ then $R[x] = RB + p$ if and only if $\bar{R}[x] = \bar{R}\bar{B} + \bar{p}$. Consequently* $\deg(p) = \deg(\bar{p})$.

Notice that if $\bar{R}$ is a division ring then $\bar{R}[x]$ is a principal ideal domain and if $\bar{p} = (\bar{f})$ then $\deg \bar{p} = D(\bar{f})$. (The symbol $D(f)$ denotes the degree of the polynomial f.)

DEFINITION 3.2. The *order* of a regular polynomial f of $R[x]$ is the minimal degree of the nonzero polynomials of $fR[x]$. The order of f is denoted by $O(f)$.

Note that $O(f) = D(f)$ when R is an integral domain.

LEMMA 3.1. *If R is completely N primary and $N^n = 0$ for a positive integer n, then $O(f)$ is equal to the exponent of the highest power of x in f whose coefficient is a unit of R.*

Proof. Let $f = a_l x^l + \ldots + a_m x^m + \ldots + a_0$ where $a_l, \ldots, a_{m+1} \in N$ and $a_m \notin N$. Certainly $O(f) \leqq m$. If $N = 0$, the theorem is obviously true. Assume inductively that the theorem is true for rings R where $N^r = 0$, $r \leqq n$. We shall show that the theorem is true for rings R where $N^n \neq 0$ and $N^{n+1} = 0$. Suppose $D(fg) < m$ where $g = b_s x^s + \ldots + b_0$, $b_s \neq 0$. Since $(R/N)[x]$ is an integral domain, we have $g \in N[x]$. Since R is completely N primary, R/N^n is completely N/N^n primary. Also, by Theorem 1.3, the coset of f is regular in $R/N^n[x]$. If $g \in N[x]$ but $g \notin N^n[x]$ then, by the induction hypothesis, $D(fg)$ is not less than m. If, on the other hand, $g \in N^n[x]$ and $D(fg) < m$ then $a_m b_s = 0$ which is impossible because a_m is a unit of R. Thus, in any case, the assumption $D(fg) < m$ leads to a contradiction. Consequently $O(f) = m$.

THEOREM 3.3. *Let R be a completely N primary ring with $N^n = 0$ for some positive integer n. If, for $f \in R[x]$, the principal ideal $(f) = fR[x] = R[x]f$ then (f) is generated by a monic polynomial of degree m if an only if $f = a_l x^l + \ldots + a_m x^m + \ldots + a_0$ where $a_l, \ldots, a_{m+1} \in N$ and a_m is a unit of R.*

([5]) A nonzero polynomial of $R[x]$ is called monic if its leading coefficient is a unit element of R.

Proof. If f is of this form let B denote the set of elements $x^{m-1}, \ldots, x, 1$ of $R[x]$. Then we may write $\bar{R}[x] = \bar{R}\bar{B} + (\bar{f})$. From Theorem 3.2 it follows that $R[x] = RB + (f)$. Thus $x^m = -b_{m-1}x^{m-1} - \ldots - b_0 + f_1$ where $f_1 \in (f)$ and $b_i \in R$, $i = 0, 1, \ldots, m-1$. Hence $g = x^m + b_{m-1}x^{m-1} + \ldots + b_0 \in (f)$. Now there exist polynomials h and k such that $f = hg + k$ where $k = 0$ or $D(k) < m$. By Lemma 3.1, $O(f) = m$ and hence $k = 0$. Thus $R[x]g = (f)$. Similarly $gR[x] = (f)$. Thus $(g) = (f)$ where g is monic.

Conversely, let $(f) = (g)$ where g is monic of degree m. Then $O(f) = O(g)$ (by the definition of $O(\ )$), while $O(g) = m$ by Lemma 3.1 (because g is monic). Thus $O(f) = m$, which means again by Lemma 3.1 that f is of the required form.

4. **Primary ideals in R, where R/N is a principal ideal domain.** If R is a principal ideal domain then the A.C.C. holds for right ideals and hence $P(q)$ is an ideal when q is a right P primary ideal of R. Moreover, in this case $P = N = 0$ and the completely prime ideals of R/N are maximal right ideals. Hence, by Definition 0.1, a principal ideal domain is an extendable ring.

If q is a P primary ideal then $P(q) = N(q)$ since $P(q)$ is a completely prime ideal. Thus $P(q)$ is also a maximal left ideal.

THEOREM 4.1. *If R is a principal ideal domain and $q = (a)$ is a P primary ideal in R with $P(q) = (b)$ then b is irreducible and $a = vb^n = b^n u$ where u and v are units of R.*

Proof. If $b = cd$ then c or d must be in (b). Suppose $d \in (b)$, say $d = eb$([6]). Then $b = ceb$ and c is a unit of R. If $c \in (b)$, say $c = be$, then $b = bed$ and d is a unit of R. Hence b is irreducible.

For the second part of the proof we have $(a) \subseteq (b)$ where b is irreducible. Let $a = bc$ where $c \in R$. If $b \notin (a)$ then $c^n \in (a)$ since (a) is P primary. Hence $c \in P((a)) = (b)$. Thus $c = bd$ and $a = b^2 d$. This process continues until $a = b^n d'$ where $b^n \in (a)$ and $b^{n-1} \notin (a)$. Consequently, $b^n = ae$ and $a = aed'$. Thus d' is a unit and $a = b^n v = ub^n$ where u and v are units of R.

LEMMA 4.1. *If R/N is a principal ideal domain and q is an ideal of R, then there is an element a in q such that $q = aR + N' = Ra + N'$ where $N' = q \cap N$.*

Proof. Let $N' = q \cap N$. Since $\bar{R}$ is a principal ideal domain $\bar{q} = (\bar{a}) = \bar{a}\bar{R} = \bar{R}\bar{a}$ for some $\bar{a} \in \bar{R}$. Let a be any element of q such that $aH = \bar{a}$. Obviously $aR + N' \subseteq q$. Moreover, if $b \in q$ then $bH = \bar{a}\bar{r}$ where $\bar{r} \in \bar{R}$. Thus $b = ar + n$ where $r \in R$ and $n \in N$. Since $n = b - ar$, n is also in q and therefore $n \in N'$. Hence $q = aR + N'$. Similarly $q = Ra + N'$.

THEOREM 4.2. *Let R be an extendable ring such that R/N is a principal ideal domain. If q is a P primary, not nil, nontrivial ideal of R then*

$$q = (v\pi^k + n)R + N' = R(v\pi^k + n) + N'$$

([6]) Note that if b is any element of a principal ideal domain R such that bR is a left, whence two-sided ideal, then from [5, p. 37] $bR = Rb$, which one may denote by (b).

where v is a unit of R, $n \in N$, π is an irreducible element of $\bar{R}$ and $N' = N \cap q$. Moreover, $P(q) = \pi R + N$ and $R/P(q)$ is a division ring.

Proof. By Lemma 4.1, $q = aR + N' = Ra + N'$ where $a \in q$. From 3b of [2] the not nil ideal $aR + N'$ is P primary in R if and only if the ideal $(\bar{a})$ is P primary in $\bar{R}$. From Theorem 4.1 we may write $\bar{a} = \bar{v}\bar{\pi}^k = \bar{\pi}^k\bar{u}$ where $\bar{\pi}$ is irreducible in $\bar{R}$ and $\bar{u}$, $\bar{v}$ are units of $\bar{R}$. Hence $a = \pi^k u + n_1 = v\pi^k + n_2$ where u and v are units of R and $n_1, n_2 \in N$.

To find $P(q)$, write $\bar{q} = (\bar{v}\bar{\pi}^k) = (\bar{\pi}^k) = \bar{\pi}^k\bar{R}$. Clearly $\bar{\pi}\bar{R} \subseteq P(\bar{q})$. Moreover, since $\bar{\pi}$ is irreducible, the ideal $\bar{\pi}\bar{R}$ is maximal in $\bar{R}$ and hence $P(\bar{q}) = \bar{\pi}\bar{R}$. From 2g of [2] we have $(P(q))^- = (\pi R)^-$ and hence $P(q) = \pi R + N$. The fact that $R/P(q)$ is a division ring is a consequence of statement 3.1 of [2].

THEOREM 4.3. *If R is a completely N primary ring which satisfies the A.C.C. for right ideals then $R[x]$ is an extendable ring, $\bar{R}[x]$ is a principal ideal domain and $R[x]$ is $N[x]$ primary.*

Proof. If R satisfies the A.C.C. then N is nilpotent and, from Theorem 1.3, $R[x]$ is $N[x]$ primary. Since $\bar{R}$ is a division ring, $\bar{R}[x]$ is a principal ideal domain([7]). It remains to show that $R[x]$ is an extendable ring. If q is a right P primary ideal in $R[x]$ then, since $R[x]$ satisfies the A.C.C., $P(q)$ is an ideal of $R[x]$ and, from Theorem 1.1, $P(R[x]) = N(R[x])$. Finally, condition (iii) of Definition 0.1 holds since $R[x]/N(R[x]) = R[x]/N[x] = \bar{R}[x]$ is a principal ideal domain.

5. **$R[x]$, where R is a completely N primary ring.** In this section let R denote a completely N primary ring which satisfies the A.C.C. for right ideals. By Theorem 4.3, $R[x]$ is an extendable ring, $N(R[x]) = N[x]$ and $(R/N)[x]$ is a principal ideal domain. In addition, by [6, p. 199], $(N[x])^t = 0$ for some integer t.

Certainly the not nil ideals of $R[x]$ are the regular ideals of $R[x]$([8]). If q is a regular ideal in $R[x]$ then, by §4, $\bar{q} = (\bar{f})$ in $\bar{R}[x]$ where f is regular in $R[x]$. Hence all the regular ideals of $R[x]$ are of the form $q = fR[x] + N'$ where f is regular in $R[x]$ and $N' = q \cap N[x]$. By §3 we have $O(f) = D(\bar{f})$.

THEOREM 5.1. *An ideal q of $R[x]$ has finite degree if and only if q is a regular ideal. In this case, $\deg(q) = O(\bar{f}) = D(\bar{f})$.*

Proof. By Theorem 3.2, q has finite degree if and only if $\bar{q}$ has finite degree, i.e., if and only if $q \nsubseteq N[x]$. Thus a necessary and sufficient condition that q have finite degree is that q be regular. Again, by Theorem 3.2, the degree of q is the same as the degree of $\bar{q} = (\bar{f})$. Since $\bar{R}$ is a division ring $D(\bar{f}) = O(\bar{f})$.

([7]) See [5, Chapter 3].

([8]) An ideal q of a ring R is called regular if it contains at least one regular element. Thus if R is N primary then it contains only regular ideals and nil ideals, for if an ideal q is not regular then every element of q is a divisor of zero and hence $q \subseteq N$.

If q is a regular, nontrivial N primary ideal of $R[x]$, then by Theorem 4.2, $q = (v(x)p(x)^k + n(x))R[x] + N'$ where $v(x)$ is a unit of $R[x]$, $\bar{p}(x)$ is an irreducible polynomial of $\bar{R}[x]$, $n(x) \in N[x]$ and $N' = N[x] \cap q$. The radical $N(q) = p(x)R[x] + N[x]$. Then $\deg(q) = k \deg(\bar{p})$ and $\bar{R}[x]/(\bar{p}(x))$ is a division ring. Thus $R[x]/q$ is a completely N primary ring.

6. **Simple algebraic extensions.** In this section let $R \subseteq A$ where R and A are completely N primary rings which satisfy the A.C.C. If $\sigma \in A$, where $a\sigma = \sigma a$ for all $a \in R$, the symbol $R[\sigma]$ shall denote the smallest subring of A containing R and σ. The symbol $R(\sigma)$ shall denote the smallest completely N primary ring containing R and σ. In the latter case, $R(\sigma)$ is called a *simple extension* of R. Certainly $R[\sigma] \subseteq R(\sigma)$ and, if x is an indeterminate, $R[\sigma]$ is the homomorphic image of the polynomial ring $R[x]$ under the homomorphism $f(x) \to f(\sigma)$. Since $R[\sigma]$ is a subring of a completely N primary ring we know that $R[\sigma]$ is P primary. The kernel of the homomorphism must then be a P primary ideal q of $R[x]$ and $R[x]/q \cong R[\sigma]$. In addition $q_* = q \cap R$ is the zero ideal since q is the set of polynomials which have σ as a root. As in §5, q is either a regular ideal or a nil ideal.

DEFINITION 6.1. Let $R \subseteq A$ and let $\sigma \in A$ where $a\sigma = \sigma a$ for all $a \in R$. If σ satisfies at least one regular polynomial of $R[x]$ then σ is called *central algebraic* with respect to R. If σ satisfies only nilpotent polynomials of $R[x]$ then σ is called *central transcendental* with respect to R([9]). The ideal q consisting of the polynomials of $R[x]$ which have σ as a root is called the *defining ideal* of σ.

We call $R(\sigma)$ a *simple algebraic extension* of R if σ is algebraic with respect to R and a *simple transcendental extension* of R if σ is transcendental with respect to R.

Let $S = R(\sigma)$ be a simple algebraic extension of R and let q be the defining ideal of σ. Then q is a not nil, nontrivial P primary ideal of $R[x]$. By §§4 and 5 we may write $q = (v(x)p(x)^k + n(x))R[x] + N'$ where the symbols have the same meanings as before. Then $R[x]/q$ is a completely N primary ring whose residue class ring is isomorphic to $\bar{R}[x]/(\bar{p}(x))$ and $R[x]/q$ is an extension of degree $kD(\bar{p}(x))$ of $R/q_* = R$. Since $R[x]/q \cong R[\sigma]$ and $R[\sigma]$ satisfies the A.C.C. we have

THEOREM 6.1. *If $S = R(\sigma)$ is a simple algebraic extension of R, the defining ideal q of σ has the form $q = (v(x)p(x)^k + n(x))R[x] + N'$ where $v(x)$ is a unit of $R[x]$, $\bar{p}(x)$ is irreducible in $\bar{R}[x]$, $n(x) \in N[x]$ and $N' = q \cap N[x]$. Then $S = R(\sigma) = R[\sigma]$ which satisfies the A.C.C. Moreover, S is a finite extension of R where $[S:R] = kD(\bar{p}(x))$. The division ring $\bar{S} = \bar{R}(\bar{\sigma})$ is obtained from $\bar{R}$ by the adjunction of the zero $\bar{\sigma}$ of the irreducible polynomial $\bar{p}(x) \in \bar{R}[x]$ and hence $[S:R] = k[\bar{S}:\bar{R}]$.*

([9]) Hereafter we shall refer to central algebraic (central transcendental) elements as algebraic (transcendental) elements.

Next we shall prove

THEOREM 6.2. *Let $S = R(\sigma)$ be a simple algebraic extension of R where $[S:R] = k[\bar{S}:\bar{R}]$. Then, S is a principal extension of R if and only if $k = 1$. For any k, $N(S) = p(\sigma)R[\sigma] + N[\sigma]$ and hence there exists a positive integer h such that $N(S)^h = 0$.*

Proof. Writing q in the form stated in Theorem 6.1 we have, as in §4, $N(q) = p(x)R[x] + N(R[x])$. Since $R[x]$ is a principal extension of R this can be written $N(q) = p(x)R[x] + N \cdot R[x] = p(x)R[x] + N[x]$. It follows from 2h of [2] that $N(R[x]/q) = N(q)/q = (p(x)R[x] + N[x])/q$. The isomorphism from $R[x]/q$ onto $S = R(\sigma)$ maps $(p(x)R[x] + N[x])/q$ onto $p(\sigma)R[\sigma] + N[\sigma]$ and hence $N(S) = p(\sigma)R[\sigma] + N[\sigma]$. Now $N(q)/q$ is a nil ideal in $R[x]/q$ and, since the A.C.C. holds, the ideal $N(q)$ of R is nilpotent modulo q. Thus there is a positive integer h such that $N(S)^h = 0$. Finally, S is a principal extension of R if and only if $p(\sigma) \in N[\sigma]$; i.e., if and only if $p(x) \in q'$, where $q' = p(x)^k R[x] + N[x]$. If $p(x) \in q'$ then q' contains a regular polynomial of degree $D(\overline{p(x)})$. However, from §3, the minimal degree of the regular polynomials in q' is $D(\overline{p(x)^k}) = kD(\overline{p(x)})$. Hence $kD(\overline{p(x)}) \leqq D(\overline{p(x)})$ and therefore $k = 1$. Conversely if $k = 1$, the extension is clearly principal.

An element σ of a ring A is called *principal* with respect to a subring R if $R(\sigma)$ is a principal extension of R. It follows from Theorem 6.2 that an algebraic element σ is principal if and only if it is a root of a nontrivial fundamental irreducible([10]) of $R[x]$.

EXAMPLE 6.1. Let R be a completely N primary ring satisfying the A.C.C. If x is an indeterminate, the ring $R[x]$ is N primary. Let q be a regular, nontrivial N primary ideal of $R[x]$ such that $q_* = 0$. As above, we have that $q = (v(x)p(x)^k + n(x))R[x] + N'$ and $R[x]/q$ is a completely N primary ring which contains R. Setting $\sigma = \bar{x}$, where $\bar{x}$ is the coset of x in $R[x]/q$, then σ is algebraic over R with defining ideal equal to q.

EXAMPLE 6.2. Let D be the division ring of quaternions with coefficients in the rational numbers and D^* the division ring of quaternions with coefficients in the real numbers. For an indeterminate x, $R = D[x]/(x^n)$ is completely N primary and is contained in the completely N primary ring $R^* = D^*[x]/(x^n)$. If $\sigma = \sqrt{2}$, then $R_1 = R(\sqrt{2}) = D_1[x]/(x^n)$ where D_1 is the ring of quaternions with coefficients in the set of all real numbers of the form $a + b\sqrt{2}$ where a and b are rational numbers. Thus R_1 is a simple algebraic extension of R of degree 2.

For an indeterminate y, the ring $R[y]$ is N primary. Then $y^2 - 2$ is a minimal degree polynominal satisfied by $\sqrt{2}$. One can use the division algorithm to show that the defining ideal of $\sqrt{2}$ is $q = (y^2 - 2)R[y]$. Then q is N primary and $R[y]/q \cong R_1$ where R_1 is an extension of R of degree 2. Similarly, we could adjoin

([10]) See [2,2e] for the meaning of this term.

to R_1 the element $\sqrt{3}$. Thus $R_2 = R_1(\sqrt{3}) = D_2[x]/(x^n)$ where D_2 is the ring of quaternions with coefficients in the set of all real numbers of the form $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$ where a, b, c and d are rational numbers. Then R_2 is an extension of R_1 of degree 2 and an extension of R of degree 4.

7. **Quotient rings.** If a ring R has a right quotient ring as described in [5, p. 118] we shall denote this ring by $Q(R)$. Then $Q(R)$ is a ring containing R such that every regular element of R has an inverse in $Q(R)$ and any element of $Q(R)$ may be written in the form $ab^{-1} = a/b$ where a, $b \in R$ and b is regular. A necessary and sufficient condition for the existence of $Q(R)$ is that for any pair of elements a, b in R, b regular, there exists a common right multiple $m = ab_1 = ba_1$ such that b_1 is regular. We shall use this criterion to establish the following theorem, which generalizes a result of A. W. Goldie [4, p. 592].

THEOREM 7.1. *Let R be a ring with identity which satisfies the A.C.C. for right ideals and suppose the elements not in $N(R/N^k)$ are regular in R/N^k for all positive integers k. Then $Q(R)$ exists.*

Proof. Since the elements not in N are regular in R, R/N is an integral domain. Hence, if $N = 0$, then by Theorem 1 of [4], $Q(R)$ exists. We proceed by induction on the smallest integer n such that $N^n = 0$. Assume that the theorem is true when $N^k = 0$ and $N^{k-1} \neq 0$. In R suppose $N^{k+1} = 0$ and $N^k \neq 0$. Then the ring $\tilde{R} = R/N^k$ satisfies the hypothesis of the theorem and, by the induction hypothesis, $Q(\tilde{R})$ exists.

Let $a \neq 0$ and $b \notin N$. If $a \in N^k$ we consider the right ideals $I_n = aR + baR + \ldots + b^n aR$, $n = 0,1,2,\ldots$. By the A.C.C., $I_t = I_{t+1}$ for some integer t and we may write $b^{t+1}a = ar_0 + bar_1 + \ldots + b^t ar_t$, $r_i \in R$. Since $b^{t+1}a \neq 0$ and $N^{k+1} = 0$, not all the r_i are in N. Let $i = h$ be the first subscript for which $r_i \notin N$. It follows that $b^{t+1-h}a = ar_h + \ldots + b^{t-h}ar_t$ and hence $b(b^{t-h}a - ar_{h+1} - \ldots - b^{t-h-1}ar_t) = ar_h$ where r_h is regular. Thus a and b have a common right multiple. If, on the other hand, $a \notin N^k$, then, since $Q(\tilde{R})$ exists, we have $\tilde{a}\tilde{c} = \tilde{b}\tilde{d}$ in $\tilde{R} = R/N^k$ where $c \notin N$. Thus $ac = bd + e$ with $e \in N^k$. If $e = 0$, we stop. Otherwise, as above, write $bf = eg$ where $g \notin N$, whence $acg = b(dg + f)$ with cg regular. Thus $Q(R)$ exists.

THEOREM 7.2. *If R is a completely N primary ring which satisfies the A.C.C. for right ideals then $R[x]$ has a right quotient ring $Q(R[x])$.*

Proof. From Theorem 1.3, for any integer n, the divisors of zero of $R/N^n[x]$ are contained in $N/N^n[x]$. Hence $R[x]$ satisfies the hypothesis of Theorem 7.1.

LEMMA 7.1. *If a ring R has a right quotient ring $Q(R)$ and if R satisfies the A.C.C. for right ideals then $Q(R)$ satisfies the A.C.C. for right ideals.*

Proof. Let k and h denote right ideals of $Q(R)$ where $k \subset h$. As in the proof of Lemma 1.3 of [3] it follows that $k_* \subset h_*$ in R. The lemma is now immediate.

LEMMA 7.2. *If a ring R with A.C.C. on right ideals has a right quotient ring $Q(R)$ and if the elements not in N are regular in R then $Q(R)$ is completely N primary.*

Proof. Let $T = \{a/b | a/b \in Q(R), a \in N\}$. For any elements a/b and c/d of T, there exist regular elements b_1 and d_1 in R such that $m = db_1 = bd_1$. Using the rule for addition in $Q(R)$ we have $a/b - c/d = (ad_1 - cb_1)/m$ which is in T since a and c are in N. Next, consider any elements a/b of T and c/d of $Q(R)$. Let c_1, $b_1 \in R$, b_1 regular, such that $cb_1 = bc_1$. Then $(a/b)(c/d) = (ac_1/db_1)$ which proves that $TQ(R) \subseteq T$. Similarly, one can prove that $Q(R)T \subseteq T$. Thus T is an ideal in $Q(R)$. It is easily seen that for any positive integer n, the product of n elements of T can be written in the form a/b where $a \in N^n$ and b is regular. Since N is nilpotent, $T \subseteq N(Q(R))$. Also, the elements of $Q(R)$ which are not in T are units and hence $Q(R)/T$ is a division ring. Then T is maximal ideal and $T = N(Q(R))$. Thus $Q(R)$ is completely N primary.

LEMMA 7.3. *If a ring R has a right quotient ring $Q(R)$ which is completely N primary then $Q(R)$ is the smallest completely N primary ring containing R.*

Proof. Let R' be a completely N' primary ring where $N' = N(R')$ and suppose $R \subseteq R'$. If $b \in R$ is regular then $b \notin N'$ and hence b has an inverse b^{-1} in R'. Thus $a, b \in R$, b regular implies $ab^{-1} \in R'$; that is, $Q(R) \subseteq R'$.

THEOREM 7.3. *Let R be a completely N primary ring which satisfies the A.C.C. for right ideals and let q be a P primary ideal of $R[x]$ with $q \subseteq N(R[x])$. Then $Q(R[x]/q)$ exists and is completely N primary. Moreover, $Q(R[x]/q)$ satisfies the A.C.C. for right ideals and is the smallest completely N primary ring containing $R[x]/q$.*

Proof. We know, by Theorem 7.2, that $Q(R[x])$ exists. Furthermore, as in the proof of Lemma 1.2 of [3], one can show that $q = q^* \cap R[x]$, where q^* is the extension of q to $Q(R[x])$. It follows that the mapping $f(x) + q \to f(x) + q^*$, $f(x) \in R[x]$, is an isomorphism of $R[x]/q$ into $Q(R[x])/q^*$. We shall identify $R[x]/q$ with the subring of $Q(R[x])/q^*$ which corresponds to $R[x]/q$ under this isomorphism.

If $f(x) + q$ is regular in $R[x]/q$ then $f(x) \notin N[x]$. By Theorem 1.3, $R[x]$ is $N[x]$ primary and consequently $f(x)$ is regular in $R[x]$. Hence $f(x)$ has an inverse $f(x)^{-1}$ in $Q(R[x])$ and we have $f(x)^{-1} + q^* = (f(x) + q^*)^{-1}$. Thus the regular elements of $R[x]/q$ have inverses in $Q(R[x])/q^*$. Now let $f(x)g(x)^{-1} + q^* \in Q(R[x])/q^*$ where $f(x), g(x) \in R[x]$, $g(x)$ regular. Then $f(x)g(x)^{-1} + q^* = (f(x) + q^*)(g(x) + q^*)^{-1}$. This proves that $Q(R[x])/q^*$ is a right quotient ring for $R[x]/q$.

The remaining part of the theorem follows from Lemmas 7.1, 7.2 and 7.3.

8. **Simple transcendental extensions.** In this section, R will always denote a completely N primary ring which satisfies the A.C.C. for right ideals and A will denote a completely N primary ring which contains R.

Let $\sigma \in A$ be transcendental over R. Then $R[\sigma]$ is an N primary ring where $N(R[\sigma]) = N(A) \cap R[\sigma]$. For an indeterminate x we have the usual homomorphism θ of $R[x]$ onto $R[\sigma]$ defined by $f(x)\theta = f(\sigma)$. The defining ideal q of σ is then a nil, P primary ideal of $R[x]$ and $q_* = q \cap R = 0$. Since $R[x]$ satisfies the A.C.C. for right ideals, q is a nilpotent ideal and $q \subseteq N(R[x]) = N[x]$ by [2, §1]. By Theorem 7.3, $Q(R[x]/q)$ exists and, since $R[\sigma] \cong R[x]/q$, $Q(R[\sigma])$ exists. Moreover, $Q(R[\sigma])$ is the smallest completely N primary ring containing R and σ. Hence $Q(R[\sigma]) = R(\sigma)$, the simple transcendental extension of R by σ. Also, $R(\sigma)$ satisfies the A.C.C. for right ideals. This establishes the first part of

THEOREM 8.1. *If $S = R(\sigma)$ is a simple transcendental extension of R then $R(\sigma) = Q(R[\sigma])$ and $R(\sigma)$ satisfies the A.C.C. for right ideals. The division ring $(R(\sigma))^- = \bar{R}(\bar{\sigma})$ is obtained by adjoining the transcendental element $\bar{\sigma}$ to $\bar{R}$. To every unit of S a unique order can be associated.*

To prove the last part of the theorem we observe from above that $(R[\sigma])^- \cong (R[x]/q)^- \cong (R/N)[x] = \bar{R}[x]$, where $\bar{R}$ is a division ring. To every nonzero element $\bar{r}$ of $(R[\sigma])^-$ a unique degree is associated, namely the degree of the polynomial of $\bar{R}[x]$ which is the image of $\bar{r}$ under the isomorphism from $(R[\sigma])^-$ onto $\bar{R}[x]$. If we extend this isomorphism to an isomorphism from $Q((R[\sigma])^-)$ onto $\bar{R}(x) = Q(\bar{R}[x])$ then to every nonzero element of $Q((R[\sigma])^-)$ a unique degree is associated, namely the degree of the corresponding (image) element of $\bar{R}(x)$. (The degree of a fraction, by definition, is the maximum of the degrees of the numerator and denominator.) Since $(Q(R[\sigma]))^- \cong Q((R[\sigma])^-)$ there is a unique degree associated with each element of $(Q(R[\sigma]))^-$. Now the set of not nilpotent elements of the completely N primary ring $Q(R[\sigma]) = R(\sigma)$ coincides with the set of units of $Q(R[\sigma])$. We define the *order* of a unit r of $R(\sigma)$ as the degree of the element $\bar{r}$ onto which r is mapped by the natural homomorphism from $Q(R[\sigma])$ onto $(Q(R[\sigma]))^-$. Thus, to each unit of $Q(R[\sigma]) = R(\sigma)$ a unique order is associated([11]).

THEOREM 8.2. *Every completely N primary ring R which satisfies the A.C.C. for right ideals is properly contained in just such a ring. Specifically, $R \subset Q(R[x])$, which is a completely N primary ring satisfying the A.C.C. for right ideals.*

Proof. See Theorem 7.2, Lemma 7.1 and Lemma 7.2.

LEMMA 8.1. *Let S be a principal extension of R and let $N(R) = P(R)$ and $N(S) = P(S)$. Let n be an ideal in $N(S)$ and suppose that $N(S/n) = N(S)/n$. Then S/n is a principal extension of R/n_*, where $n_* = n \cap R$.*

([11]) Note that for σ algebraic we have $R[\sigma] = R(\sigma)$ and hence in this case it is also true that $Q(R[\sigma]) = R(\sigma)$.

Proof. As in [2, §2], we may assume that $\tilde{R} = R/n_* \subseteq S/n$ where, for any set $B \subseteq S$, $\tilde{B}$ denotes the image of B under the natural homomorphism from S to S/n. Let v be an element of S such that $\tilde{v} \in N(S/n) = N(S)/n$. Then $v \in N(S)$ and, since S is a principal extension of R we may write $v = \sum v_i\sigma_i$ where $v_i \in N(R)$ and $\sigma_i \in S$. Hence $\tilde{v} = \sum \tilde{v}_i\tilde{\sigma}_i$ where $\tilde{v}_i \in N(R/n_*)$ and $\tilde{\sigma}_i \in S/n$. Thus S/n is a principal extension of R/n_*.

THEOREM 8.3. *If S is a simple transcendental extension of R then S is a principal extension of R.*

Proof. Let $S = R(\sigma)$ and let n denote the defining ideal of the transcendental element σ. Applying Lemma 8.1 to $R[x]$ and R, and noting that $n_* = n \cap R = 0$, we have that $R[x]/n$ is a principal extension of $R/n_* = R$. Thus $R[\sigma] = R[x]/n$ is a principal extension of R. Now $Q = Q(R[\sigma])$ is a principal extension of $R[\sigma]$ since $N(Q)$ consists of elements of the form $a/b = ab^{-1}$ where $a \in N(R[\sigma])$ and $b^{-1} \in Q(R[\sigma])$. Hence $N(Q) = N(R[\sigma]) \cdot Q(R[\sigma]) = N(R) \cdot R[\sigma] \cdot R(\sigma) = N(R) \cdot R(\sigma)$, i.e., $Q(R[\sigma]) = R(\sigma)$ is a principal extension of R.

EXAMPLE 8.1. Let R be a completely N primary ring which satisfies the A.C.C. for right ideals. Let x be an indeterminate and let n be any nil, P primary ideal of $R[x]$ such that $n_* = 0$. Setting $\sigma = \bar{x}$, where $\bar{x}$ is the coset of x in $R[x]/n$, then σ is transcendental over R with defining ideal n.

EXAMPLE 8.2. Let R and R^* be as in Example 6.2 and let F be the field of rational numbers. Then for the transcendental number π we have $R(\pi) = D'[x]/(x^n)$ where D' is the ring of quaternions with coefficients of the form $p(\pi)/q(\pi)$ where $p(\pi)$ and $q(\pi)$ are elements of $F[\pi]$, $q(\pi) \neq 0$.

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